IV. Factor Models and Arbitrage Pricing Theory

1. Introduction

- CAPM prices securities by using a general equilibrium approach
- Securities can also be priced using the condition of absence of arbitrage in an efficient market (the no–arbitrage approach)
  - Arbitrage is a strategy that brings a riskless profit at zero cost
- No–arbitrage approach is called arbitrage pricing theory (APT)
- The APT does not require very strong assumptions like the CAPM does. It also provides an additional insight into the multifaceted nature of systematic risk
- First we examine factor models that predate the APT by several years.

2. Factor Models

- Factor models were introduced to simplify the computations of the Markowitz portfolio selection model
- Recall that to estimate the efficient frontier we needed:
  - n estimates of expected returns, n estimates of variances and, n(n - 1)/2 estimates of covariances
  - For a relatively small portfolio of 50 stocks that would mean 1,325 estimates
- Besides that, statistical errors in estimation of correlation coefficients can lead to results that do not make sense (i.e., negative portfolio variance)

- It is known that covariances between securities tend to be positive because the same economic forces (such as business cycle, inflation, money-supply changes) affect most firms in the economy

- Let us group all relevant factors into one macroeconomic indicator and assume that it moves the security market as a whole

- Assume that beyond the common movements due to macroeconomic indicator all the uncertainty in stock returns is firm-specific

- We recognize that different firms have different sensitivities to macroeconomic events. Then

$$r_i = E(r_i) + \beta_i F + e_i \tag{4-1}$$

  where F is the unanticipated component of the macro factor, β_i is the responsiveness of security i to macro-events and e_i is the impact of unanticipated firm–specific events

- Equation (4-1) is known as a **single-factor model** for stock returns

- We emphasize that the only contribution of a factor model lies in its assumption that return of a single stock is *linearly* affected to systematic and non–systematic sources of risk

- A common proxy for the macro factor is the market index, such as TSE 300; then equation (4-1) is called a **single–index model**.

- A single–index model is often written as following

  – Let $E(r_i) = \alpha_i + r_f + \beta_i[E(r_M) - r_f]$ and $F = r_M - E(r_M)$, then

$$\begin{aligned} r_i &= E(r_i) + \beta_i F + e_i = \alpha_i + r_f + \beta_i[E(r_M) - r_f] + \beta_i[r_M - E(r_M)] + e_i \\ &= \alpha_i + r_f + \beta_i(r_M - r_f) + e_i \end{aligned}$$

  – Or,

$$R_i = \alpha_i + \beta_i R_M + e_i,$$

  where

    * $R_i = r_i - r_f$, $R_M = r_M - r_f$ are excess returns and
    * α_i is the stock's expected return if the market excess return $r_M - r_f$ is zero

- Let us show that the above β_i is the same as in CAPM
- To find β_i, let us take the covariance of the both sides of equation $R_i = \alpha_i + \beta_i R_M + e_i$ with R_M:

$$Cov(R_i, R_M) = Cov(\alpha_i, R_M) + Cov(\beta_i R_M, R_M) + Cov(e_i, R_M)$$

  or $Cov(R_i, R_M) = \beta_i \sigma_M^2$ since $Cov(\alpha_i, R_M) = Cov(e_i, R_M) = 0$. Therefore,

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2}$$

- Note that the factor model does not give us the value for α_i

• The found result allows us to solve a portfolio selection problem with much less computations. Indeed, let us find variance and covariance of returns

$var(R_i) = var(\alpha_i + \beta_i R_M + e_i) = var(\beta_i R_M + e_i) = var(\beta_i R_M) + var(e_i) = \beta_i^2 var(R_M) + var(e_i) = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$,

where we took into account that

- α_i, β_i are constant and
- covariance between R_M and e_i is zero by definition

$Cov(R_i, R_j) = Cov(\alpha_i + \beta_i R_M + e_i,\ \alpha_j + \beta_j R_M + e_j) = Cov(\beta_i R_M + e_i,\ \beta_j R_M + e_j) = Cov(\beta_i R_M, \beta_j R_M) + Cov(e_i, e_j) = \beta_i \beta_j Cov(R_M, R_M) = \beta_i \beta_j \sigma_M^2$

where we took into account that

- covariance between e_i and e_j is zero by definition

• Now to estimate the efficient frontier we need only:

- n estimates of expected returns, $E(r_i)$
- n estimates of sensitivity coefficients, β_i
- n estimates of firm-specific variances, $\sigma^2(e_i)$
- 1 estimate of the variance of the macroeconomic factor, σ_M^2

- So for a portfolio of 50 stocks we would need only 151 estimates instead of 1,325 when a single index model is not used
- The trade off here is that factor model has to be correct. That is, we must know the factor and it has to affect returns linearly
- Investors find β_i, α_i as well as $\sigma^2(e_i)$ by using a linear regression analysis.

Index Model and Diversification

- Index model offers insight into portfolio diversification
- We know that systematic risk cannot be diversified away, no matter how many securities you add to the portfolio
- Let us calculate this non-diversifiable risk for a portfolio
- Suppose we choose an equally weighted portfolio of n securities
- According to the index model, the excess return on each security is given by

$$R_i = \alpha_i + \beta_i R_M + e_i$$

- Let us multiply both sides of the last equation by $\frac{1}{n}$ and sum all equations over i:

$$R_p = \alpha_p + \beta_p R_M + e_p,$$

where

$$\alpha_p = \frac{1}{n} \sum_{i=1}^{n} \alpha_i$$

$$\beta_p = \frac{1}{n}\sum_{i=1}^{n}\beta_i$$

$$e_p = \frac{1}{n}\sum_{i=1}^{n}e_i$$

- Hence the portfolio's variance is

$$\sigma_p^2 = \beta_p^2\sigma_M^2 + \sigma^2(e_p)$$

- The systematic risk component of the portfolio variance, $\beta_p^2\sigma_M^2$ will persists regardless on the extent of diversification

- In contrast, the nonsystematic component of the portfolio variance, $\sigma^2(e_p)$ will become negligible with more stocks in the portfolio:

$$\sigma^2(e_p) = var(\frac{1}{n}\sum_{i=1}^{n}e_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2(e_i) = \frac{1}{n}\overline{\sigma^2(e)},$$

where $\overline{\sigma^2(e)}$ is the average of the firm–specific variances. Since this average is independent of n, $\frac{1}{n}\overline{\sigma^2(e)}$ approaches zero with increasing n

Practice Problem

If the beta of your well-diversified portfolio is 0.5 and standard deviation of market return is 15%, what is the minimum standard deviation you can achieve?

Multifactor Models

- Confining systematic risk to a single factor is not compelling (recall that the sources of systematic risk include uncertainty about business cycle, interest rate, inflation and so on)

- In particular, many risky securities have sensitivities to various sources of systematic risk that are different than their sensitivities the the broad market index. For example, security X can be more sensitive to inflation risk than it is to the market risk. However, if we assume that $r_X = \alpha_X + r_f + \beta_X(r_M - r_f) + e_X$ then its sensitivity to the inflation would be the same as it is to the market

- Therefore, **multifactor models** can provide better description of returns

- The return of stock i according to multifactor model is given by

$$r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i1}F_1 + ... + \beta_{in}F_n + e_i,$$

where F_k, $(k = 1, ..., n)$ is the unanticipated component of macro factor k, β_{ik} is the responsiveness of security i to macro factor k, and e_i is the impact of unanticipated firm–specific events

- Factors can represent GDP change, inflation rate, interest rate ...

- Unfortunately, the multifactor model does not tell us what factors should be used and how to find $E(r_i)$. It simply assumes that there are n macro–factors that affect the returns linearly.

3. Arbitrage Pricing Theory

- Factor model cannot predict α

- Like the CAPM, the APT predicts a security market line linking expected return to risk. That is, it predicts that α should be zero

- The APT relies on three propositions:

  - (1) Security returns can be described by a factor model

    * In this section we assume that single factor model holds

- (2) There is a sufficient number of securities to diversify away idiosyncratic risk
- (3) There are no arbitrage opportunities

- Let's introduce a definition of arbitrage:
  - Arbitrage is a zero cost strategy that yields positive profit without any downside risk

- To create a zero investment portfolio without any exposure to risk of loosing money, you will need to buy and sell an equal dollar amount of assets carrying the same risk.

- To receive a profit, you will have to short sell an overpriced asset and buy an underpriced one;

Example of Arbitrage

Suppose there are two well-diversified portfolios, A and B, in the economy; portfolio A has $\beta_A = 1$ and expected return of 10%, while the portfolio B has $\beta_B = 0.5$ and the expected return of 5%. The risk-free rate is 4%.

- First, let us determine which portfolio, if any, is mispriced.

- From one–factor model:

$$r_A = E(r_A) + \beta_A F = 0.10 + F$$

$$r_B = E(r_B) + \beta_B F = 0.05 + 0.5 \times F$$

Let us compose a portfolio C with weight in riskless asset of $w_f = 0.5$ and that in asset A of $w_A = 0.5$, then

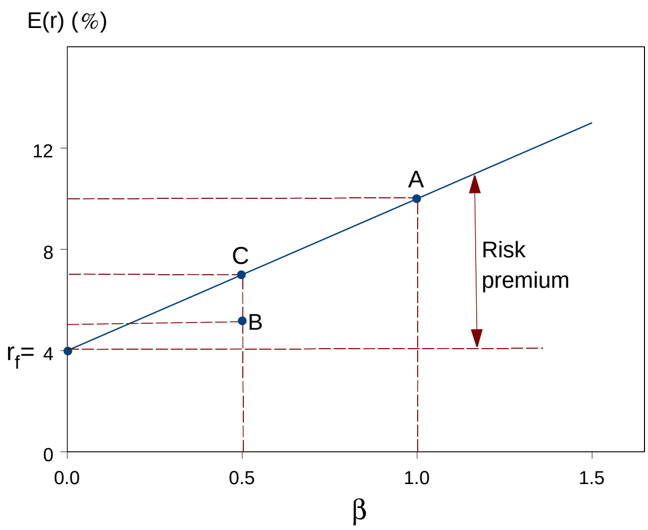
$$r_C = w_f r_f + w_A r_A = w_f r_f + w_A[E(r_A) + \beta_A F] = 0.5 \times 0.04 + 0.5(0.10 + F) = 0.07 + 0.5 \times F$$

- It follows that asset B and portfolio C provides the same risk exposure (since $\beta_B = \beta_C$), but return of asset B is lower.

- We conclude that asset B is overpriced with respect to asset A and an investor can follow the following arbitrage strategy:
  - Sell short asset B and use all the proceeds to buy portfolio C.
  - The cost of this strategy is zero but the profit is positive and riskless. For example, if we short asset B worth $\$1,000,000$ and invest this amount into portfolio C, then our return will be:

$$\begin{array}{c} -\$1,000,000(1 + 0.05 + 0.5 \times F) \text{ (short position in B)} \\ \$1,000,000(1 + 0.07 + 0.5 \times F) \text{ (long position in C)} \\ \hline \$1,000,000(0.02) = \$20,000 \end{array}$$

- As people start short–selling portfolio B, its price will decline and expected return will increase until the arbitrage opportunity is eliminated
- It follows that assumption of APT that arbitrage opportunities are absent is reasonable
- The following figure shows the expected return–beta relationships for the above assets and portfolio C



- Because portfolios A and C and T-bills do not allow arbitrage, we expect from this example that to preclude arbitrage opportunities, the expected return on all well-diversified portfolios must lie on the straight line from the risk-free asset. That is, for any asset the ratio $\frac{E(r)-r_f}{\beta}$ must be the same for the absence of arbitrage opportunities in economy.

Practice Problem

There are two well–diversified portfolios in the economy, portfolio A and portfolio B. Portfolio A has beta of 1.5 and expected return of 15%, while portfolio B has beta 0.9 and expected return of 9.5%. If the risk–free rate is 5%, is there an arbitrage opportunity? If so, show your arbitrage strategy.

APT Derivation

- Now let us show that the line on the graph above is the SML
- First, let us prove the conclusion from the last example formally. That is we

want to show that to preclude arbitrage opportunities, the expected return on all well diversified portfolios must lie on the straight line from the risk-free asset

- Suppose that two well-diversified portfolios, U and V, are combined into a zero-beta portfolio, Z, with the following weights

$$w_U = \frac{\beta_V}{\beta_V - \beta_U} \quad \text{and} \quad w_V = -\frac{\beta_U}{\beta_V - \beta_U}$$

- Note that the weights sum up to one and that the portfolio beta is zero:

$$\beta_Z = w_U \beta_U + w_V \beta_V = \frac{\beta_V}{\beta_V - \beta_U}\beta_U - \frac{\beta_U}{\beta_V - \beta_U}\beta_V = 0$$

- Portfolio Z is riskless:
  - it has no firm-specific risk, because ...
  - it has no exposure to systematic risk, because ...

- We assume absence of arbitrage opportunities in the market. Therefore, portfolio Z has to earn only risk-free rate of return

$$E(r_Z) = w_U E(r_U) + w_V E(r_V) = \frac{\beta_V}{\beta_V - \beta_U} E(r_U) - \frac{\beta_U}{\beta_V - \beta_U} E(r_V) = r_f$$

rearranging

$$\frac{E(r_U) - r_f}{\beta_U} = \frac{E(r_V) - r_f}{\beta_V} \tag{4-2}$$

- We conclude that risk premiums should be proportional to betas

- Question: What goes wrong in the proof above if portfolios U and V are not well–diversified?

Practice Problem

Consider the one-factor APT. Assume that two portfolios, A and B, are well diversified. The betas of portfolios A and B are 0.5 and 1.5, respectively.

The expected returns on portfolios A and B are 12% and 24%, respectively. Assuming no arbitrage opportunities exist, what must be the risk-free rate?

- Now suppose one of the portfolios, say U is the market portfolio. Then the expression (4-2) simplifies to (since $\beta_M = 1$)

$$E(r_V) = r_f + \beta_V[E(r_M) - r_f]$$

- We have used a no arbitrage argument to obtain expected return-beta relationship that is identical to CAPM

- In our derivation of the above expected return–beta relationship we avoided the restrictive assumptions of CAPM

- Note that APT does not require that the benchmark portfolio be the true market portfolio; any well-diversified portfolio on the SML may serve as a benchmark portfolio. If the benchmark portfolio is U then

$$E(r_V) = r_f + \beta_V \frac{E(r_U) - r_f}{\beta_U}$$

Question: Do we have to know the market portfolio to use the equation above?

- However, CAPM predicts how the market risk premium (and the risk premium of an individual security) depends on the average risk aversion of investors, while APT cannot provide any insight into the microeconomics of the market risk premium

Individual Asset and the APT

- We have derived the APT for well–diversified portfolios. What about an individual asset?

- It can be shown that the APT holds for *almost* all individual securities. Indeed, if no–arbitrage expected return–beta relationship holds for infinitely many different, well–diversified portfolios, it must be virtually certain that the relationship holds for all but a small number of individual securities

- We conclude that advantages of the APT come at price: while the CAPM provides the expected return–beta relationship for all securities, the APT cannot rule out the violation of this relationship for any particular asset

- Along the same line, if we find a single security with a non–zero α we will not be able to get advantage of mispricing by creating an arbitrage strategy. In this regard, APT is worthless.

A Multifactor APT

- If return is given by a multifactor model, then we can derive a multifactor APT.

- Again, we assume that there are enough securities to diversify the systematic risk and arbitrage opportunities are absent in the market

- As an illustration of this derivation let us assume that asset returns are captured by a two–factor model so that return of a single security i:

$$r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$$

- For example, F_1 can stand for the departure of GDP growth from expectations, and F_2 is an unanticipated change in interest rate

- For a well–diversified portfolio P:

$$r_P = E(r_P) + \beta_{P1}F_1 + \beta_{P2}F_2$$

- To establish a multifactor APT, let us first introduce a **factor portfolio** which is a well–diversified portfolio constructed to have a beta of 1 on one of the factors and a beta of 0 on any other factor.

  – Return on a factor portfolio for factor k (k=1,2):

$$r_k = E(r_k) + F_k$$

- The expected return of a factor portfolio is equal to the expected return of the corresponding factor (e.g., think of the inflation rate as being a factor then factor portfolio for inflation has expected return equal to inflation rate)

- The multifactor APT states that the overall risk premium on a well–diversified portfolio must equal the sum of risk premiums required as compensation for each source of systematic risk:

$$E(r_P) = r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f], \tag{4-3}$$

  where $E(r_1)$ and $E(r_2)$ are the expected returns on the first and second factor portfolios, respectively

- Since $E(r_1)$ and $E(r_2)$ are typically known from empirical analysis, one does not have to construct factor portfolios in practice. The latter are typically used as a theoretical tool

- If the last expression does not hold then arbitrage opportunities arise in economy:

  – Form a competing portfolio Q by investing in factor portfolios with the following weights: β_{P1} in the first factor portfolio, β_{P2} in the second factor portfolio, and $1 - \beta_{P1} - \beta_{P2}$ in T–bills so

$$r_Q = \beta_{P1}[E(r_1) + F_1] + \beta_{P2}[E(r_2) + F_2] + (1 - \beta_{P1} - \beta_{P2})r_f$$

$$= r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f] + \beta_{P1}F_1 + \beta_{P2}F_2$$

- Portfolios Q and P have the same risk
- The expected return of portfolio Q

$$E(r_Q) = r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$$

- Suppose that equation (4-3) does not hold and $E(r_P) > r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$
- Because $r_P > r_Q$ we short portfolio Q and use the proceeds to buy portfolio P. Because both portfolios have the same risk exposure, we end up with positive profit at no risk. The cost of this strategy is zero.

  For example, if we short portfolio Q worth $\$1,000,000$ then the profit is $\$1,000,000\Bigg(E(r_P) - \Bigg[r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]\Bigg]\Bigg)$
- Similar result is found if $E(r_P) < r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$

Practice Problem

Suppose that the market can be described by the following three sources of systematic risk with associated risk premiums

Factor	Risk Premium of a factor portfolio (%)
Industrial Production	6
Interest rates	2
Consumer confidence	4

The return on a particular well–diversified portfolio is generated according to the following equation

$$r_P = 15\% + 1.0I + 0.5R + 0.75C,$$

where I, R and C are unanticipated components in Industrial Production, Interest rates and Consumer confidence, respectively.

i. Find the efficient rate of return of this portfolio using the APT. The T–bill rate is 6%. Is the stock over– or underpriced? Explain.

ii. Does the market allow an arbitrage strategy? If it does, then find a strategy.

Testing APT empirically

- Empirical testing of APT is complicated since it requires identifying factors and finding expected returns
- APT provides a better fit to data than CAPM studied in Topic 3.
- If multifactor CAPM is used with the same factors then APT provides the same results. However, multifactor CAPM gives a guidance as to where to look for those factors.

Additional Practice Problems

1. Suppose you held a well-diversified portfolio with a very large number of securities, and that the single index model holds. If the σ of your portfolio was

0.20 and σ_M was 0.16, the β of the portfolio would be approximately

A) 0.64

B) 0.80

C) 1.25

D) 1.56

E) none of the above

2. Consider the single factor APT. Portfolio A has a beta of 0.2 and an expected return of 13%. Portfolio B has a beta of 0.4 and an expected return of 15%. The risk-free rat of return is 10%. If you wanted to take advantage of an arbitrage opportunity, you should take a short position in portfolio and a long position in portfolio

A) A, A

B) A, B

C) B, A

D) B, B

E) none of the above

3. Consider the multifactor APT with two factors. Stock A has an expected return of 16.4%, a beta of 1.4 on factor 1 and a beta of .8 on factor 2. The risk premium on the factor 1 portfolio is 3%. The risk-free rate of return is 6%. What is the risk-premium on factor if no arbitrage opportunities exit?

Use the following information to answer questions 4 and 5:

Consider the multifactor APT. There are two independent economic factors, F_1 and F_2. The risk-free rate of return is 6%. The following information is available about two well-diversified portfolios:

Portfolio	β on F_1	β on F_2	Expected Return (%)
A	1.0	2.0	19
B	2.0	0.0	12

4. Assuming no arbitrage opportunities exist, the risk premium on the factor F_1 portfolio should be

A) 3%

B) 4%

C) 5%

D) 6%

E) none of the above

5. Assuming no arbitrage opportunities exist, the risk premium on the factor F_2 portfolio should be

A) 3%

B) 4%

C) 5%

D) 6%

E) none of the above